MA6251 - MATHEMATICS II

REGULATION 2013

ANSWER KEY

TIME : 3 HOURS **QP CODE: 50776** MAX. MARKS : 100

PART – A

		Marks
1.	$\frac{\partial(x+3y)}{\partial x}+\frac{\partial(y-2z)}{\partial y}+\frac{\partial(x+2kz)}{\partial z}=0 \Rightarrow 1+1+2k=0 \Rightarrow k=-1.$	2
2.	$\int_c \bar{F}.d\vec{r}=\int_c xy^2\,dx+(x^2+y^2)dy=\int_2^4(5x^4-12x)dx=228$	2
3.	$\frac{1}{D^2-6D+9}3log2.e^{0x}=\frac{1}{0-0+9}3log2=\frac{log2}{3}$	2
4.	$\theta(\theta-1)+4\theta+2=0 \Rightarrow m^2-3m+2=0 \Rightarrow m=-1,-2$ $\Rightarrow y=Ae^{-z}+Be^{-2z} where\ x=e^z \Rightarrow y=\frac{A}{x}+\frac{B}{x^2}.$	2
5.	$L[f(t)]=\int_0^4 e^{-st}e^{-t}dt=\left[\frac{e^{-(s+1)t}}{-(s+1)}\right]_{t=0}^{t=4}=\frac{1-e^{-4(s+1)}}{s+1}.$	2
6.	$f(\infty)=\underset{s\to0}{\text{lt}}\ sL[f(t)]=\underset{s\to0}{\text{lt}}\ s\frac{1}{s(s+\alpha)}=\frac{1}{\alpha}.$	2
7.	$u_x=e^x cosy, u_{xx}=e^x cosy\ \&u_y=1-e^x siny, u_{yy}=-e^x cosy$ $\Rightarrow u_{xx}+u_{yy}=0 \Rightarrow u\ is\ harmonic.$	2
8.	$w=z^2=x^2-y^2+i(2xy)=u+iv\ \&\ x=1 \Rightarrow u=1-y^2\&v=2y$ $\Rightarrow u=1-\left(\frac{v}{2}\right)^2 \Rightarrow v^2=4(1-u)$ *is a parabola in w – plane.*	2
9.	$f(z)=\frac{z-1}{z+1}=\frac{u}{u+2}=\frac{u}{2}\left\{1+\frac{u}{2}\right\}^{-1}=\frac{u}{2}-\left\{\frac{u}{2}\right\}^2+\left\{\frac{u}{2}\right\}^3+\cdots$ if $\lvert u\rvert\le 1\&\ u=z-1.$	2
10.	$\underset{z\to0}{\text{lt}}\frac{sinz}{z}=\underset{z\to0}{\text{lt}}\frac{cosz}{1}=\frac{1}{1}=1$ *L'H rule* $\Rightarrow z=0$ *is an* ~~*essential*~~ removable *singularity.*	2

PART – B

(5 X 16 = 80)

11. (a)(i) 8

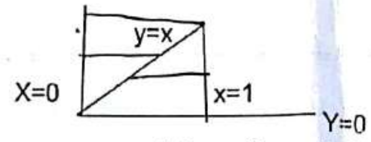
$$\nabla X\bar{F}=\begin{vmatrix} i & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-y^2+x & -2xy-y & 0\end{vmatrix}=0 \Rightarrow \bar{F}\ is\ conservative.$$ 2

$\bar{F}=\nabla\phi \Rightarrow \frac{\partial\phi}{\partial x}=x^2-y^2+x, \frac{\partial\phi}{\partial y}=-2xy-y\ \&\frac{\partial\phi}{\partial z}=0$ 2

Integrating, $\phi=\frac{x^3}{3}-y^2x+\frac{x^2}{2}+c_1, \phi=-y^2x+\frac{y^2}{2}+c_2\ \&\ \phi=c_3$ 2

Hence, the scalar potential function= $\phi=\frac{x^3}{3}-y^2x+\frac{x^2}{2}+\frac{y^2}{2}+C.$ 2

(ii) 8



2

Limits are : $x=y, x=1\ \&\ y=0, y=1.$ *Given* $P=xy-x^2, Q=x^2y$ 2

$\int_C \bar{F}.d\vec{r}=\int_C Pdx+Qdy=\iint_R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=\int_0^1\int_1^y(2xy-x)dxdy=-\frac{1}{12}$ 4

(OR)

$\int_0^1\int_y^1(2xy-x)\,dxdy$

(b)(i) $\phi = x^2 + y^2 - 16, \nabla\phi = 2x\vec{i} + 2y\vec{j}, \hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{x\vec{i}+y\vec{j}}{4}$ 1 8

$\bar{F}.\hat{n} = \left(z\vec{i} + x\vec{j} - 3y^2x\vec{k}\right).\left(\frac{x\vec{i}+y\vec{j}}{4}\right) = \frac{zx+yx}{4}$ 2

Projection of S on yz plane: $dS = \frac{dydz}{|\hat{n}.\vec{i}|} = \frac{dydz}{\frac{x}{4}}$ 1

$\iint_S \bar{F}.\hat{n}\,dS = \iint_S \bar{F}.\hat{n}\frac{dydz}{|\hat{n}.\vec{i}|} = \iint_S \bar{F}.\hat{n}\frac{dydz}{\frac{x}{4}} = \int_0^5\int_0^4 \frac{zx+yx}{4}.\frac{dydz}{\frac{x}{4}}$

$= \int_0^5\int_0^4 (z+y)dydz = 90.$ 4

(ii) $\nabla.\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial x} + \frac{\partial F_3}{\partial x} = 2(x+y+z)$ 2 8

by Gauss Divergence theorem,

$\iint_S \bar{F}.\hat{n}\,dS = \iiint_V \nabla.\bar{F}\,dv = \int_0^1\int_0^1\int_0^1 2(x+y+z)dxdydz = 3.$ 2+4

12. (a)(i) $m^3 - 5m^2 + 7m - 3 = 0 \Rightarrow m = 1,1,3\ \&\ CF = (A+Bx)e^x + Ce^{3x}$ 3 8

$PI = \frac{1}{(D-3)(D-1)^2}\left(\frac{e^{3x}+e^x}{2}\right) = \frac{1}{2}\frac{1}{(D-3)(D-1)^2}e^{3x} + \frac{1}{2}\frac{1}{(D-3)(D-1)^2}e^x$

$= \frac{1}{2}e^{3x}\frac{1}{(D-3).4}(1) + \frac{1}{2}e^x\frac{1}{(-2)(D-1)^2}(1) = \frac{1}{8}xe^{3x} - \frac{1}{4}.\frac{x^2}{2}e^x$ 3

$y(x) = (A+Bx)e^x + Ce^{3x} + \frac{xe^{3x}-x^2e^x}{8}.$ 2

(ii) Eliminating y, $(D^2 + 4D - 5)x = -6e^{2t}$ *where* $D = \frac{d}{dt}$ 2 8

$m = -5,1, CF = Ae^{-5t} + Be^t, PI = \frac{1}{D^2+4D-5}(-6e^{2t}) = -\frac{6}{7}e^{2t}$ 2

$x(t) = Ae^{-5t} + Be^t - \frac{6}{7}e^{2t}.$ 2

$y(t) = -\frac{1}{3}\frac{dx}{dt} - \frac{2}{3}x = Ae^{-5t} - Be^t + \frac{8}{7}e^{2t}.$ 2

(OR)

(b)(i) $1 + x = e^z\ \&z = \log(x+1), (x+1)D = \theta, (x+1)^2D^2 = \theta(\theta-1)$ 1 8

$(\theta^2+1)y = sin2z, CF = Acosz + Bsinz, PI = \frac{sin2z}{-3}$ 1+2+2

$y(z) = Acosz + Bsinz - \frac{sin2z}{3}$ *where* $z = \log(x+1)$ 1

$y(z) = Acos\log(x+1) + Bsin\log(x+1) - \frac{sin2(\log(x+1))}{3}.$ 1

(ii) $m^2 - 6m + 9 = 0 \Rightarrow m = 3,3\ \&CF = Ae^{3x} + Bxe^{3x} = Ay_1 + By_2$ 2 8

$\Delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = e^{6x}, A(x) = -\int\frac{RHS.y_2}{\Delta}dx + c_1 = -logx + c_1,$ 2

$B(x) = \int\frac{RHS.y_1}{\Delta}dx + c_2 = -\frac{1}{x} + c_2$ 2

$y(x) = c_1e^{3x} + c_2xe^{3x} - e^{3x}(1 + logx).$ 2

13. (a)(i) $(i) I = \int_0^\infty e^{-2t}t\,sin3tdt = [L(tsin3t)]_{t=2} = -\frac{d}{ds}\left\{\frac{3}{s^2+9}\right\}_{s=2} = \frac{12}{169}.$ 2+2 8

$(ii) L^{-1}\left\{cot^{-1}\left(\frac{2}{s+1}\right)\right\} = -\frac{1}{t}L^{-1}\left\{\frac{d}{ds}\left(cot^{-1}\left(\frac{2}{s+1}\right)\right)\right\}$

$= -\frac{1}{t}L^{-1}\left\{\frac{2}{(s+1)^2+4}\right\}$

$= -\frac{1}{t}e^{-t}L^{-1}\left\{\frac{2}{s^2+4}\right\} = -\frac{e^{-t}sin2t}{t}.$ 2+2

(ii) $L^{-1}[f(s).g(s)] = \int_0^t f(u)g(t-u)du = \frac{1}{a^2}\int_0^t sinau\,sina(t-u)du$ 8

$= \frac{1}{2a^2}\int_0^t (\cos(2au - at) - cosat)du$

$= \frac{1}{2a^2}\left[\frac{\sin(2au-at)}{2a} - ucosat\right]_0^t$

$= \frac{sinat - atcosat}{2a^2}.$ 3+3+2

 (OR)

(b)(i) $L\{f(t)\} = \frac{1}{1-e^{-as}}\left\{\int_0^{\frac{a}{2}}(E)e^{-st}\,dt + \int_{\frac{a}{2}}^{a}(-E)e^{-st}\,dt\right\}$ 8

$= \frac{E}{s}\cdot\frac{1-2e^{-\frac{sa}{2}}+e^{-sa}}{1-e^{-as}}$

$= \frac{E}{s}\cdot\frac{1-e^{-\frac{sa}{2}}}{1+e^{-\frac{sa}{2}}} = \frac{E}{s}\tanh\frac{as}{4}.$ 3+3+2

(ii) $L\{y''+y'\} = L\{t^2+2t\}$ 8

$s^2L(y(t)) - sy(0) - y'(0) + sL(y(t)) - y(0) = \frac{2}{s^3}+\frac{2}{s^2}$

$(s^2+s)L(y(t)) - 4s + 2 - 4 = \frac{2}{s^3}+\frac{2}{s^2}$

$L(y(t)) = \frac{2}{s^4}+\frac{2}{s}+\frac{2}{s+1} = \frac{2}{3!}t^3 + 2t + 2e^{-t} = 2\left(\frac{t^3}{6}+t+e^{-t}\right).$ 1+2+3+2

14. (a)(i) $z = re^{i\theta}, f(z) = z^n = r^n(\cos n\theta + i\sin n\theta) = u + iv$ 8

$u = r^n\cos n\theta \Rightarrow\ u_r = nr^{n-1}\cos n\theta\ \&\ u_\theta = -nr^n\sin n\theta$

$v = r^n\sin n\theta \Rightarrow\ v_r = nr^{n-1}\sin n\theta\ \&\ v_\theta = nr^n\cos n\theta$

$u_r = \frac{1}{r}v_\theta = nr^{n-1}\cos n\theta\ \&\ u_\theta = -\frac{1}{r}v_r = nr^{n-1}\sin n\theta$

$\Rightarrow f(z) = z^n$ *is analytic for all n.* 1+2+2+2+1

(ii) $\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$ 2 8

$\frac{(w-i)(0+i)}{(w+i)(0-i)} = \frac{(z-1)(i+1)}{(z+1)(i-1)} \Rightarrow w = \frac{1+iz}{1-iz}$ 2

$w = \frac{1+iz}{1-iz}$ or $z = i.\left\{\frac{1-w}{1+w}\right\}.$ 1

$|z| < 1 \Rightarrow \left|i.\left\{\frac{1-w}{1+w}\right\}\right| < 1 \Rightarrow |1-w| < |1+w|$ 1

$\Rightarrow ((1-u)-iv)^2 < ((1+u)-iv)^2$ 1

$\Rightarrow 1 - 2u + u^2 + v^2 < 1 + 2u + u^2 + v^2 \Rightarrow u > 0.$ 1

(OR)

(b)(i) $u = \log\sqrt{x^2+y^2} = \frac{1}{2}\log(x^2+y^2) \Rightarrow u_x = \frac{x}{x^2+y^2}, u_y = \frac{y}{x^2+y^2},$ 1 8

$u_{xx} = \frac{y^2-x^2}{(x^2+y^2)^2}, u_{yy} = \frac{x^2-y^2}{(x^2+y^2)^2} \Rightarrow u_{xx}+u_{yy} = 0 \Rightarrow u$ *is harmonic.* 2

$f(z) = \int(u_x(z,0) - i\,u_y(z,0))dz + c = \int\left(\frac{1}{z} - i(0)\right)dz + c = \log z + c$ 2

$f(z) = \log(x+iy) + c = \log re^{i\theta} + c = \log\sqrt{x^2+y^2} + i\tan^{-1}\left(\frac{y}{x}\right) + c$

$= u + iv$ 2

The conjugate harmonic of $u = v = \tan^{-1}\left(\frac{y}{x}\right) + c.$ 1

(ii) $w = \frac{1}{z}\ \&\ z = \frac{1}{w} = \frac{u}{u^2+v^2} + i\frac{-v}{u^2+v^2} = x + iy \Rightarrow x = \frac{u}{u^2+v^2}\ \&y = \frac{-v}{u^2+v^2}.$ 8

$|z-2i| = 2 \Rightarrow |x + i(y-2)| = 2 \Rightarrow x^2 + y^2 = 4y$

$\left(\frac{u}{u^2+v^2}\right)^2 + \left(\frac{-v}{u^2+v^2}\right)^2 = 4\left(\frac{-v}{u^2+v^2}\right) \Rightarrow 1 = -4v \Rightarrow 1 + 4v = 0.$ 3+3+2

15. (a)(i) 8

$z = \pm 1$ *are inside C, the rectangle having vertices* $2 \pm i, -2 \pm i$

$I = \int_C \frac{\cos\pi z}{z^2-1} dz = \int_c \left[\frac{1}{2}.\frac{1}{z-1} - \frac{1}{2}.\frac{1}{z+1}\right] \cos\pi z dz = \frac{1}{2}\int_c \frac{\cos\pi z}{z-1} dz - \frac{1}{2}\int_c \frac{\cos\pi z}{z-(-1)} dz$

$I = \frac{1}{2}.2\pi i \cos\pi(1) - \frac{1}{2}.2\pi i \cos\pi(-1) = -\pi i + \pi i = 0.$ 1+1+4+2

(ii) $I = \int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \int_{C:|z|=1} \frac{\frac{dz}{iz}}{2+\frac{z^2+1}{2z}} = \frac{2}{i}\int_c \frac{1}{z^2+4z+1} dz$ 8

$= \frac{2}{i}\int_c f(z)dz = \frac{2}{i}(2\pi i \, x \text{ sum of the residues of } f(z) \in C)$

$z^2 + 4z + 1 = 0 \Rightarrow z = -2 \pm \sqrt{3} \Rightarrow |z| = |-2+\sqrt{3}| < 1 \;\&\; |z| = |-2-\sqrt{3}| > 1$

$z = -2+\sqrt{3}$ is inside c : let $\alpha = -2+\sqrt{3}$ *and* $\beta = -2-\sqrt{3}$

Residue at $(z = \alpha)$ = $\text{lt}_{z\to\alpha}(z-\alpha)\frac{1}{(z-\alpha)(z-\beta)} = \frac{1}{\alpha-\beta} = \frac{1}{2\sqrt{3}}$.

Therefore, $I = \frac{2}{i}.2\pi i.\frac{1}{2\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$ 3+1+1+2+1

(OR)

(b)(i) $f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$ 2 8

$(i)|z| < 1$ *and* $\left|\frac{z}{2}\right| < 1$

$f(z) = -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - (1-z)^{-1} = -\frac{1}{2}\left[1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \cdots\right] - [1 + z + z^2 + \cdots]$

$f(z) = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n + \sum_{n=0}^{\infty} z^n = \text{Taylor series about } z = 0$ 3

$(ii) 1 < |z| < 2 \Rightarrow \left|\frac{1}{z}\right| < 1 \;\&\; \left|\frac{z}{2}\right| < 1$

$f(z) = -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} = -\frac{1}{2}\left[1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \cdots\right] - \frac{1}{z}\left[1 + \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 + \cdots\right]$

$f(z) = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n+1} = \text{Laurent series about } z = 0$ 3

(ii) $I = \int_0^{\infty} \frac{1}{(x^2+a^2)(x^2+b^2)} dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)(x^2+b^2)} dx$ 8

$= \frac{1}{2}\int_{-R}^{R} \frac{1}{(z^2+a^2)(z^2+b^2)} dz + \frac{1}{2}\int_{\frown} \frac{1}{(z^2+a^2)(z^2+b^2)} dz = \frac{1}{2}\int_c f(z)dz = \frac{1}{2}2\pi i(R_1 + R_2).$ 3

$R_1 at\ z = ia\ (order\ 1) = \text{lt}_{z\to\alpha}(z-ia)\frac{1}{(z^2+a^2)(z^2+b^2)} = -\frac{1}{2ia(a^2-b^2)}$ 2

$R_2 at\ z = ib\ (order\ 1) = \text{lt}_{z\to\alpha}(z-ib)\frac{1}{(z^2+a^2)(z^2+b^2)} = +\frac{1}{2ib(a^2-b^2)}$ 2

$I = \frac{1}{2}.2\pi i(R_1 + R_2) = \frac{1}{2}.2\pi i\left(-\frac{1}{2ia(a^2-b^2)} + \frac{1}{2ib(a^2-b^2)}\right) = \frac{\pi}{2ab(a+b)}.$ 1
