MA6351 - TRANSFORMS AND PARTIAL DIFFERENTIAL EQUATIONS

 REGULATION 2013

ANSWER KEY

1. $z = f(x^2-y^2)$, $p = \frac{\partial z}{\partial x} = 2x f'(x^2-y^2)$, $q = \frac{\partial z}{\partial y} = -2y f'(x^2-y^2)$ (1+1) = (2M)

 $\Rightarrow yp + xq = 0$

2. This is form $F(p,q)=0$. The complete integral is (1M)

$z = ax + (1-\sqrt{a})^2 y + c$ (1M)

3. The function $f(x)$ of period 2π in $(c, c+2\pi)$ can be expanded as $f(x) = \frac{a_0}{2} + \sum_{n\geq 1} a_n \cos nx + \sum_{n\geq 1} b_n \sin nx$ if the following conditions are satisfied $(4\times\frac{1}{2})$ = (2M)

i) $f(x)$ is periodic with period 2π and bounded in $(c, c+2\pi)$

ii) $f(x)$ is piecewise continuous with finite no of discontinuities in $(c, c+2\pi)$

iii) $f(x)$ has atmost a finite no of maxima and minima in $(c, c+2\pi)$

4. $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}$, where $c_n = \frac{1}{2l}\int_{-l}^{l} f(x) e^{-in\pi x/l} dx$ (1+1) = (2M)

5. Solution of 1-dimensional wave eqn. $y_{tt} = a^2 y_{xx}$ is (1M)

$y(x,t) = (A\cos \lambda x + B\sin \lambda x)(C\cos \lambda a t + D\sin \lambda a t)$ which is periodic w.r.to time t.

Solution of 1-dimensional heat eqn $u_t = a^2 u_{xx}$ is (1M)

$u(x,t) = (A\cos \lambda x + B\sin \lambda x)e^{-\lambda^2 a^2 t}$ which is non-periodic w.r.to time t.

6. Two solutions involving exponential terms are (1+1M)

i) $u(x,y) = (A\cos \lambda x + B\sin \lambda x)(Ce^{\lambda y} + De^{-\lambda y})$

ii) $u(x,y) = (Ae^{\lambda x} + Be^{-\lambda x})(C\cos \lambda y + D\sin \lambda y)$

7. $F\{f(ax)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax) e^{isx} dx = \frac{1}{a} F\left(\frac{s}{a}\right)$ (1+1M)

8. If $F[f(x)] = F(s)$ and $F\{g(x)\} = G(s)$, then (2M)

$F\{f(x) * g(x)\} = F(s)\cdot G(s)$

9. $Z\left[\frac{1}{n}\right] = \sum_{n=1}^{\infty} \frac{1}{n} z^{-n} = \log_e\left(\frac{z}{z-1}\right), |z|>1$ (2M)

10. $y_n = a\cdot 2^n \Rightarrow y_{n+1} = 2a\cdot 2^n \therefore \frac{y_{n+1}}{y_n} = 2 \Rightarrow y_{n+1} - 2y_n = 0$ (1+1M)

11(a) i) The given eqn is Clairauts form (1M)

The complete integral is $z = ax + by + a^2 - b^2$ (1M)

Diff. the above w.r.to a and b, we get ...... (2M)

$a = -x/2$, $b = y/2$ ...... (3M)

$\therefore z = \frac{1}{4}(y^2 - x^2) \Rightarrow 4z = y^2 - x^2$

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ii) The given eqn. is a Lagrange's linear eqn with $P = x - 2z$, $Q = 2z - y$ and $R = y - x$ (1M)

The subsidiary eqns are $\dfrac{dx}{x-2z} = \dfrac{dy}{2z-y} = \dfrac{dz}{y-x}$ ... Eqn (1) ...... (1M)

Using the multipliers 1, 1, 1, each ratio in Eqn (1) $= \dfrac{dx+dy+dz}{0}$

$\therefore dx + dy + dz = 0$ (2½M)

Integrating, we get $x + y + z = a$, say

Using the multipliers $y, x, 2z$, each ratio in Eqn

$= \dfrac{y\,dx + x\,dy + 2z\,dz}{0}$ (2½M)

$\therefore d(xy) + 2z\,dz = 0$

Integrating, we get $xy + z^2 = b$, say

The general solution / integral is $\phi(x+y+z,\ xy+z^2) = 0$. (1M)

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(OR)

b) i) A.E : $m^2 + 2m + 1 = 0 \Rightarrow m = -1, -1$ ...... (1M)

C.F : $x f_1(y-x) + f_2(y-x)$ ...... (1M)

$P.I_1 = \dfrac{1}{(D+D')^2} e^{x-y} = \frac{x^2}{2} e^{x-y}$ ...... (2M)

$P.I_2 = \dfrac{1}{(D+D')^2} xy = \dfrac{1}{D^2}\left(1 + \dfrac{D'}{D}\right)^{-2} xy = \dfrac{1}{D^2}\left(1 - \dfrac{2D'}{D} + \ldots\right)xy$

$= \dfrac{x^3 y}{6} - \dfrac{x^4}{12}$ ...... (3M)

$\therefore C.S = C.F + P.I \Rightarrow z = x f_1(y-x) + f_2(y-x) + \frac{x^2}{2} e^{x-y} + \frac{x^3}{12}(2y - x)$ (1M)

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ii) A.E : $m^2 - 5m + 6 = 0 \Rightarrow m = 2, 3$ ...... (1M)

C.F : $f_1(y+2x) + f_2(y+3x)$ ...... (2M)

$P.I = \dfrac{1}{(D-2D')(D-3D')} y \sin x$ ...... (1M)

$= \dfrac{1}{D-2D'}\left[\int (a-3x)\sin x\,dx\right]_{a \to y+3x} = \dfrac{1}{D-2D'}\left[-y\cos x - 3\sin x\right]$

$= -\left[\int \left[(a-2x)\cos x + 3\sin x\right]dx\right]_{a \to y+2x}$

$= 5\cos x - y\sin x$ (3M)

$\therefore C.S = C.F + P.I$

$z = f_1(y+2x) + f_2(y+3x) + 5\cos x - y\sin x$ ...... (1M)

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2(a)(i) $f(x) = x + x^2$ is neither odd nor even in $(-\pi, \pi)$. (1M)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 2\pi^2/3$$ (1M)

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 4\frac{(-1)^n}{n^2}$$ (1½ M)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = -\frac{2(-1)^n}{n}$$ (1½ M)

Fourier series of $f(x)$ is (1M)

$$f(x) = \pi^2/3 + 4\sum_{n\geq 1}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n\geq 1}\frac{(-1)^n}{n}\sin nx$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \pi^2/6$$, since $x = \pi$ is a point of discontinuity for $f(x)$, the sum of Fourier series when $x = \pi$ is $\frac{1}{2}[f(-\pi^+) + f(\pi^-)] = \pi^2$ (2M)

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ii) Fourier series upto the first harmonic is

$$y = \frac{a_0}{2} + a_1\cos x + b_1\sin x$$, where (1M)

$$a_0 = \frac{2}{6}\sum y = \frac{1}{3}\times(8.7) = 2.9$$ (2M)

$$a_1 = \frac{2}{6}\sum y\cos x = \frac{1}{3}\times(-1.1) = -0.367$$ (2M)

$$b_1 = \frac{2}{6}\sum y\sin x = \frac{1}{3}\times(0.524) = 0.175$$ (2M)

$$\therefore y = 1.45 - 0.367\cos x + 0.175\sin x$$ (1M)

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b) i) $f(x) = x,\ 0 \leq x \leq \pi$

Half-range cosine series for $f(x)$ is given by (1M)

$$f(x) = \frac{a_0}{2} + \sum_{n\geq 1} a_n\cos nx$$, where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \pi$$ (1M)

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \begin{cases} -\frac{4}{\pi n^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even}\end{cases}$$ (3M)

$$\therefore f(x) = \pi/2 - \frac{4}{\pi}\sum_{n=1,3,\ldots}\frac{1}{n^2}\cos nx$$ (1)

Since $x = 0$ is an end pt of $[0, \pi]$ for $f(x)$, sum of Fourier Cosine Series when $x = 0$ is 0 (2M)

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \pi^2/8$$

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ii) $f(x) = \begin{cases} l - x, & 0 < x \leq l \\ 0, & l \leq x < 2l\end{cases}$

(iv) Fourier series of $f(x)$ in $(0, 2l)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n\geq 1} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n\geq 1} b_n \sin\left(\frac{n\pi x}{l}\right), \text{ where} \quad \text{(1M)}$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = l/_2 \quad \text{(1M)}, \quad a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \begin{cases} \frac{2l}{n^2\pi^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \quad \text{(3M)}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx = \frac{l}{n\pi}. \quad \text{(2M)}$$

$$\therefore f(x) = l/_4 + \frac{2l}{\pi^2}\sum_{n=1,3,\ldots} \frac{1}{n^2}\cos\left(\frac{n\pi x}{l}\right) + \frac{l}{\pi}\sum_{n\geq 1}\frac{1}{n}\sin\left(\frac{n\pi x}{l}\right) \quad \text{(1M)}$$

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3 (a) The suitable solution of the equation $y_{tt} = \alpha^2 y_{xx}$ is

$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda\alpha t + D\sin\lambda\alpha t)$ Eqn (1) ... (2M)

B.c's are (i) $y(0,t) = 0$, (ii) $y(l,t) = 0 \;\forall\, t \geq 0$ ... (2M)
(iii) $y_t(x,0) = 0$, (iv) $y(x,0) = kx(l-x)$, $0 \leq x \leq l$

Use cond. (i), (ii) and (iii) in Eqn (1), we get

$A = 0$, $\lambda = \frac{n\pi}{l}$ and $D = 0$ (2+2+2M)

The most general solution is

$y(x,t) = \sum_{n\geq 1} B_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi\alpha t}{l}\right)$ Eqn (2) ... (1M)

Use cond. (iv) in Eqn (2) and half-range sine series, we get Euler's formula $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

$$= \begin{cases} \frac{8kl^2}{n^3\pi^3} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} = B_n \quad \text{(4M)}$$

$$\therefore y(x,t) = \frac{8kl^2}{\pi^3}\sum_{n=1,3,\ldots}\frac{1}{n^3}\sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi\alpha t}{l}\right). \quad \text{(1M)}$$

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(b) 1-dimensional heat eqn. is

$0^\circ c$, $100^\circ c$, A, B, $x=0$, $x=l$ ... (1M)

$$u_t = \alpha^2 u_{xx}$$

In steady state, u is independent of t and ... (2M)

$$u(x) = \frac{100}{l}x, \quad 0 \leq x \leq l$$

In unsteady state, the suitable solution is

$u(x,t) = (A\cos\lambda x + B\sin\lambda x)\,e^{-\alpha^2\lambda^2 t}$ Eqn (1) ... (2M)

B.c's are
(i) $u(0,t) = 0$, (ii) $u(l,t) = 0 \;\forall\, t \geq 0$ ... (2M)
(iii) $u(x,0) = \frac{100}{l}x$, $0 \leq x \leq l$

Apply cond. (i) and (ii) in Eqn(1), we get

$$A = 0, \quad \lambda = \frac{n\pi}{l}$$ ..... (2+2 M)

The most general solution is

$$u(x,t) = \sum_{n\geq 1} B_n \sin\left(\frac{n\pi x}{l}\right) \cdot e^{-\frac{a^2 n^2 \pi^2}{l^2} t} \quad \text{Eqn (2)}$$ .. (1M)

Use cond. (iii) and half-range sine series, we get

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx = \frac{200}{n\pi}(-1)^{n+1} = B_n$$ .. (3M)

$$\therefore u(x,t) = \frac{200}{\pi}\sum_{n\geq 1} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{a^2 n^2 \pi^2}{l^2} t}$$ (1M) 16

4(a)(i) By defn, $F_c\{f(x)\sin ax\} = \sqrt{(2/\pi)}\int_0^\infty f(x) \sin ax \sin sx \, dx$ .. (1M)

$$= \frac{1}{2}\left\{\sqrt{2/\pi}\int_0^\infty \cos(s-a)x \, dx - \sqrt{2/\pi}\int_0^\infty f(x)\cos(s+a)x \, dx\right\}$$.(2M)

$$= \frac{1}{2}\left[F_c(s-a) - F_c(s+a)\right].$$ ..... (1M) 4

ii) Fourier transform of $f(x)$ is

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) e^{isx} dx = \sqrt{2/\pi}\left(\frac{1-\cos s}{s^2}\right) = \sqrt{2/\pi}\,\frac{2\sin^2(s/2)}{s^2}$$ ..(2+4+1M)

By Parseval's identity, ... (2M)

$$\int_{-\infty}^{\infty} |F\{f(x)\}|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

$$8/\pi \int_0^\infty \frac{\sin^4(s/2)}{s^4} dx = \int_0^1 (1-x)^2 dx$$ ..... (1M)

$$\int_0^\infty \frac{\sin^4 t}{t^4} dt = \pi/3$$

 (2M) 12

(b) $f(x) = e^{-x}$

Fourier sine transform of $f(x)$ is $F_s(s) = \sqrt{2/\pi}\int_0^\infty f(x) \sin sx \, dx$ .. (1M)

$$= \sqrt{2/\pi}\left(\frac{s}{s^2+1}\right)$$ .... (3M)

By Parseval's identity for the Fourier sine transform,

$$\int_0^\infty |F_s(s)|^2 ds = \int_0^\infty |f(x)|^2 dx$$ .... (1M)

$$\therefore \int_0^\infty \frac{x^2}{(x^2+1)^2}\,dx = \pi/4 . \quad \ldots \ldots (3M)$$

Fourier cosine transform of $f(x)$ is

$$F_c(s) = \sqrt{2/\pi}\int_0^\infty f(x)\cos sx\,dx = \sqrt{2/\pi}\left(\frac{1}{s^2+1}\right) \ldots (1+3M)$$

By Parseval's identity for the Fourier cosine transform, (1M)

$$\int_0^\infty |F_c(s)|^2\,ds = \int_0^\infty |f(x)|^2\,dx$$

$$\int_0^\infty \frac{1}{(x^2+1)^2}\,dx = \frac{\pi}{4} \quad \ldots \ldots (3M)$$

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5(a)i) Let $f(n) = \frac{2n+3}{(n+1)(n+2)} = \frac{1}{n+1} + \frac{1}{n+2}$, after resolve into partial fraction. (3M)

$$Z\left[\frac{1}{n+1}\right] = z\log_e\left(\frac{z}{z-1}\right), \quad Z\left[\frac{1}{n+2}\right] = z^2\log_e\left(\frac{z}{z-1}\right) - z \quad (2+2M)$$

$$\therefore Z[f(n)] = (z^2+z)\log_e\left(\frac{z}{z-1}\right) - z . \quad \ldots \ldots (1M)$$

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(ii) $$Z^{-1}\left[\frac{z^2}{(z-1/2)(z-1/4)}\right] = Z^{-1}\left[\frac{z}{z-\frac{1}{2}}\right] * Z^{-1}\left[\frac{z}{z-\frac{1}{4}}\right] = \left(\tfrac{1}{2}\right)^n * \left(\tfrac{1}{4}\right)^n \ldots (1+2M)$$

$$= \left(\tfrac{1}{4}\right)^n * \left(\tfrac{1}{2}\right)^n \quad \ldots \ldots (1M)$$

$$= \sum_{m=0}^{n} \left(\tfrac{1}{4}\right)^m \left(\tfrac{1}{2}\right)^{n-m} \quad (2M)$$

$$= 2\left[\left(\tfrac{1}{2}\right)^n - \left(\tfrac{1}{2}\right)^{2n+1}\right] \text{ for } n = 0,1,2,\ldots \quad (2M)$$

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b(i) $$Z^{-1}\left[\frac{z^3}{(z-1)^2(z-2)}\right] = Z^{-1}[F(z)], \text{ where } F(z) = \frac{z^3}{(z-1)^2(z-2)}$$

$$\therefore \frac{F(z)}{z} = \frac{z^2}{(z-1)^2(z-2)} = \frac{(-3)}{z-1} + \frac{(-1)}{(z-1)^2} + \frac{(4)}{z-2}, \text{ resolve into partial fraction} \ldots (3M)$$

$$\therefore Z^{-1}[F(z)] = -3Z^{-1}\left[\frac{z}{z-1}\right] - 1\,Z^{-1}\left[\frac{z}{(z-1)^2}\right] + 4Z^{-1}\left[\frac{z}{z-2}\right] \quad (1M)$$

$$= (-3)1^n - n + 4\cdot 2^n \text{ for } n = 0,1,2\ldots \quad (2M)$$

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ii) $$Z[y(n+2)] - 7Z[y(n+1)] + 12Z[y(n)] = Z[2^n] \quad \ldots (1M)$$

$$\Rightarrow (z^2 - 7z + 12)F(z) = \frac{z}{z-2}, \text{ where } F(z) = Z[y(n)] \quad \ldots (3M)$$

$$\therefore \frac{F(z)}{z} = \frac{1}{(z-2)(z-3)(z-4)} = \frac{(1/2)}{z-2} + \frac{(-1)}{z-3} + \frac{(1/2)}{z-4}, \text{ resolve into partial fraction } (3M)$$

$$\therefore y_n = Z^{-1}[F(z)] = \tfrac{1}{2}\cdot 2^n - 3^n + \tfrac{1}{2}\,4^n \text{ for } n = 0,1,2,\ldots \quad (3M)$$

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